JOURNAL OF APPROXIMATION THEORY 12, 213–229 (1974)

Jackson Type Theorems for Approximation with Side Conditions

DARELL J. JOHNSON*

University of California, Riverside, California 92502

Communicated by Oved Shisha

1. INTRODUCTION

The classical Weierstrass Theorem [12] states that one may approximate a function in $C[a, b]$ arbitrarily closely in sup norm by a polynomial. The classical Jackson Theorem [7] refines the Weierstrass Theorem by obtaining quantitative rates of convergence by polynomials to a continuous function.

In this paper we obtain Jackson type results for two settings in which Weierstrass theorems already exist. We first consider Yamabe's theorem [18], which goes back to Walsh [16], and has been extended more recently by Deutsch [2] and Singer [13].

THEOREM (YAMABE). *Let M be a dense convex subset of a real normed linear space X, and suppose that $\{x_i^*\}_{i=1}^n \subseteq X^*$. Then, for each $x \in X$ and $\epsilon > 0$, there is an $m \in M$ such that $\| x - m \| < \epsilon$ and $x_i^*(m) = x_i^*(x)$ $(i = 1,..., n)$.*

In Section 2 we state and prove a Jackson Theorem version of Yamabe's theorem, which we call the bounded linear functional theorem.

The second case we treat is the so-called SAIN approximation problem, in which one requires the additional condition $\| m \| = \| x \|$ in the conclusion of Yamabe's theorem. This problem had its genesis in a result due to Wolibner [17]. Wolibner's result was generalized by Deutsch and Morris [3–5], who also gave the name SAIN to this type of approximation problem. More recently, McLaughlin and Zaretzki [11], Holmes and Lambert [6], and Lambert [9, 10] have contributed to the still incomplete characterizations obtained by Deutsch and Morris [4].

In Section 3 we consider the slightly relaxed condition $\| m \| \leqslant \| x \|$, which we term "weak SAIN" approximation, and obtain some Jackson type

* Now at the Massachusetts Institute of Technology, Cambridge, Massachusetts 02139.

theorems for normed linear spaces. In Section 4 we quickly specialize to function spaces $C(T)$, T compact Hausdorff, and obtain a general Jackson type theorem for SAIN approximation when the bounded linear functionals are all point evaluations. We also observe that one is naturally led to the open question of considering arbitrary restricted range approximation [15] in place of norm or weak norm preservation.

Although more general results than those which follow may be established (see [8]), for simplicity we have assumed in this paper that we are approximating from closed subspaces only.

2. The Bounded Linear Functional Theorem

We let X be an arbitrary normed linear space, and consider an increasing sequence of closed linear subspaces $\{M_k\}_{k=1}^{\infty}$ of X whose union M is dense in X. We suppose that $\{x_i^*\}_{i=1}^{n} \subseteq X^*$ and let x be an arbitrary fixed element of X. We let $\delta_k(x) = \delta(x; M_k)$ denote the deviation of the element x from the subspace M_k. Without loss of generality in the following, the linear functionals x_i^* may always be assumed to be linearly independent.

THEOREM 2.1. *There exist a constant C and a positive integer N such that for every x in X and each $k \geqslant N$ there is an $m_k \in M_k$ satisfying*

(1) $x_i^*(m_k) = x_i^*(x)$ $(i = 1,..., n)$,

(2) $\| x - m_k \| \leqslant C\, \delta_k(x)$.

Proof. Choose $r_1 ,..., r_n$ in M such that $x_i^*(r_j) = \delta_{ij}$. Choose N so that $r_j \in M_N$ $(j = 1,..., n)$ and set $C = 2(1 + \sum_1^n \| x_j^* \| \| r_j \|)$. Let $x \in X$ and $k \geqslant N$. Choose $s_k \in M_k$ such that $\| x - s_k \| \leqslant 2\delta_k(x)$ and set

$$m_k = s_k + \sum_{j=1}^{n} x_j^*(x - s_k)\, r_j ,$$

then $m_k \in M_k$, $x_i^*(m_k) = x_i^*(x)$ $(i = 1,..., n)$, and

$$\begin{aligned} \| x - m_k \| &\leqslant \| x - s_k \| + \sum_{j=1}^{n} | x_j^*(x - s_k)| \, \| r_j \| \\ &\leqslant \| x - s_k \| + \sum_{j=1}^{n} \| x_j^* \| \| r_j \| \| x - s_k \| \\ &= \tfrac{1}{2} C \| x - s_k \| \leqslant C\delta_k(x). \end{aligned}$$

Q.E.D.

3. Weak SAIN Approximation

We now consider the situation in which the constraint $\| m_k \| \leqslant \| x \|$ is added to the interpolating constraints in the bounded linear functional theorem. We observe that if we have no interpolating side conditions imposed, then the result is straightforward.

THEOREM 3.1. *For each x in X and $k \geqslant 1$ there exists $m_k \in M_k$ such that*

(1) $\| m_k \| \leqslant \| x \|$

(2) $\| x - m_k \| \leqslant 3\delta_k(x)$.

Proof. If $x \in M_k$, choose $m_k = x$. If $x \in X \backslash M_k$, choose $s_k \in M_k$ so that $\| x - s_k \| \leqslant \frac{3}{2}\delta_k(x)$. If $\| s_k \| \leqslant \| x \|$, take $m_k = s_k$. If $\| s_k \| > \| x \|$, let $m_k = \lambda s_k$, where λ is any number satisfying

$$\max\{0, 1 - (3/2 \| s_k \|) \delta_k(x)\} \leqslant \lambda \leqslant \| x \| / \| s_k \|.$$

Then $m_k \in M_k$, $\| m_k \| \leqslant \| x \|$, and

$$\begin{aligned} \| x - m_k \| = \| x - s_k + (1 - \lambda) s_k \| &\leqslant \| x - s_k \| + (1 - \lambda) \| s_k \| \\ &\leqslant \tfrac{3}{2}\delta_k(x) + \tfrac{3}{2}\delta_k(x) = 3\delta_k(x). \end{aligned}$$

Q.E.D.

The constant 3 in Theorem 3.1 may actually be replaced by any constant strictly bigger than 2 (see [8]).

If we have nonempty interpolatory conditions together with weak norm preservation to satisfy, the theory is no longer as simple, and in general one does not even have a Weierstrass Theorem (see [4] or Example 3.4 below). However, the following theorem gives a sufficient condition on the bounded linear functionals involved.

THEOREM 3.2. *Suppose there is an $m \in M$ such that $\| m \| < \| x \|$ and $x_i^*(m) = x_i^*(x)$ $(i = 1,..., n)$. Then there exist a constant C and a positive integer N such that for every $k \geqslant N$ there is an $m_k \in M_k$ satisfying*

(1) $x_i^*(m_k) = x_i^*(x)$ $(i = 1,..., n)$

(2) $\| m_k \| < \| x \|$

(3) $\| x - m_k \| \leqslant C\delta_k(x)$.

Proof. By the BLF Theorem there are C_1 and N_1 such that for every $k \geqslant N_1$ there exists $r_k \in M_k$ with

$$x_i^*(r_k) = x_i^*(x) \qquad (i = 1,..., n), \qquad \text{and} \qquad \| x - r_k \| \leqslant C_1\delta_k(x).$$

Let $\alpha = C_1 \| m \| [\| x \| - \| m \|]^{-1}$ (so $\| m \| = \alpha(C_1 + \alpha)^{-1} \| x \|$) and let $C = 3C_1 + 2\alpha$. Since $\delta_k(x) \to 0$, we can choose an $N_2 \geqslant N_1$ such that

$\alpha\delta_k(x) \leqslant \| x \|$ for $k \geqslant N_2$. Choose $N \geqslant N_2$ so that $m \in M_N$. Given any $k \geqslant N$, define

$$\lambda_k = \frac{(C_1 + \alpha)\,\delta_k(x)}{\| x \| + C_1\delta_k(x)} \qquad \text{and} \qquad m_k = \lambda_k m + (1 - \lambda_k)\, r_k\,.$$

Then $m_k \in M_k$, $x_i^*(m_k) = x_i^*(x)$ $(i = 1,..., n)$,

$$\begin{aligned}
\| m_k \| &\leqslant \lambda_k \| m \| + (1 - \lambda_k)\| r_k \| \\
&\leqslant \lambda_k \frac{\alpha}{C_1 + \alpha} \| x \| + (1 - \lambda_k)[\| x \| + C_1\delta_k(x)] \\
&= \frac{\alpha\delta_k(x)\| x \|}{\| x \| + C_1\delta_k(x)} + \| x \| - \alpha\delta_k(x) \\
&\leqslant \alpha\delta_k(x) + \| x \| - \alpha\delta_k(x) = \| x \|,
\end{aligned}$$

and

$$\begin{aligned}
\| x - m_k \| &= \| \lambda_k(x - m) + (1 - \lambda_k)(x - r_k)\| \\
&\leqslant \lambda_k \| x - m \| + (1 - \lambda_k)\, C_1\delta_k(x) \\
&\leqslant \frac{(C_1 + \alpha)\,\delta_k(x)}{\| x \| + C_1\delta_k(x)}\, 2 \, \| x \| + \frac{[\| x \| - \alpha\delta_k(x)]}{\| x \| + C_1\delta_k(x)}\, C_1\delta_k(x) \\
&\leqslant \left[\frac{C_1 2 \| x \| + \alpha 2 \| x \| + \| x \| C_1}{\| x \|}\right] \delta_k(x) = C\delta_k(x).
\end{aligned}$$

Q.E.D.

Remark. We observe [1, p. 38, Theorem 3] that the condition in Theorem 3.2,

(A) $\exists m \in M \ni x_i^*(m) = x_i^*(x)$ $(i = 1,..., n)$ and $\| m \| < \| x \|$ is equivalent to the condition

(B) $\exists \epsilon > 0$ such that $| \sum_{i=1}^{n} \alpha_i x_i^* x | \leqslant (\| x \| - \epsilon)\| \sum_{i=1}^{n} \alpha_i x_i^* \|$ holds for all $\alpha = (\alpha_1 ,..., \alpha_n) \in R^n$.

Since (B) holds automatically in the case $n = 1$ for any nonextremal bounded linear functional, we have the following as an immediate corollary:

THEOREM 3.3. *Suppose* $| x^*(x)| < \| x^* \| \| x \|$. *Then there exist a constant* C *and a positive integer* N *such that for all* $k \geqslant N$ *there is an* $m_k \in M_k$ *for which*

(1) $x^*(m_k) = x^*(x)$,

(2) $\| m_k \| \leqslant \| x \|$,

(3) $\| x - m_k \| \leqslant C\delta_k(x)$.

While Theorem 3.3 is not especially satisfying, it is best possible in two senses. First, one need not have SAIN (and hence not weak SAIN [4, p. 358, Lemma 2.3]) for one extremal bounded linear functional [4, p. 359, Remark 2.2], and second one need not have SAIN for two nonextremal bounded linear functionals [4, p. 359, Proposition 2.1]. For better results we must impose stronger hypotheses. Even if we consider $C[a, b]$ and polynomials, however, by modifying an example of Deutsch and Morris [4, p. 366, Remark 4.3] we can exhibit two nonextremal bounded linear functionals for which one does not have SAIN.

Example 3.4. We let $X = C[0, 1]$, and $M = \mathscr{P}$, where $\mathscr{P}$ is the set of polynomials on [0, 1]. Let

$$x_1^* = \int_0^1 dx, \qquad x_2^* = \int_{1/2}^1 dx, \qquad x_3^* = \int_0^{1/2} dx,$$

and let

$$f(x) = \begin{cases} 1, & \text{if } 0 \leqslant x \leqslant 1/2, \\ 2 - 2x, & \text{if } 1/2 \leqslant x \leqslant 1. \end{cases}$$

Then $\|f\| = \|x_1^*\| = 1$, $\|x_2^*\| = \|x_3^*\| = \frac{1}{2}$, and $x_3^* = x_1^* - x_2^*$. In particular, if $p \in \mathscr{P}$, $\|p\| = 1$, is such that $x_1^*p = x_1^*f = \frac{3}{4}$, $x_2^*p = x_2^*f = \frac{1}{4}$, then $x_3^*p = \frac{1}{2}$, so that $\|p\| \leqslant \|f\| = 1$ implies $p = 1$. Thus, $\|f - p\| = 1$ and one does not have SAIN for x_1^* and x_2^* on $C[a, b]$ with the polynomials as the dense subspace.

However, we observe that for $X = C[a, b]$ and $M = \mathscr{P}$, if $\{x_i^*\}_{i=1}^n$ are all nonextremal point evaluations, then condition (B) holds trivially (or in any case by [4, p. 362, Lemma 4.1]). Hence we have a second immediate corollary to Theorem 3.2.

THEOREM 3.5. *Suppose $X = C(T)$, with T compact Hausdorff, and let $f \in C(T)$. Suppose $\{x_i^*\}_{i=1}^n = \{e_{x_i}\}_{i=1}^n$ are point evaluations on $C(T)$ such that $|f(x_i)| < \|f\|$ $(i = 1,..., n)$, then there exist C and N such that for every $k \geqslant N$, there is an $m_k \in M_k$ for which*

(1) $m_k(x_i) = f(x_i)$ $(i = 1,..., n)$,

(2) $\|m_k\| \leqslant \|f\|$,

(3) $\|f - m_k\| \leqslant C\delta_k(x)$.

4. SAIN APPROXIMATION

We now consider the situation in which equality holds in the constraint $\|m_k\| \leqslant \|x\|$ dealt with in section three. First we treat the case without interpolatory side conditions.

THEOREM 4.1. *For each x in X there is an integer N so that for every $k \geqslant N$ there exists $m_k \in M_k$ with $\| m_k \| = \| x \|$ and $\|x - m_k\| \leqslant 4\delta_k(x)$.*

Proof. The result is trivial if $x \in M_N$ for some N. Thus we may assume $x \notin M_k$ for every k. In particular, $0 < \delta_k(x) \leqslant \| x \|$ for every k. Choose N so that $2 \| x \|^{-1} \delta_k(x) < 1$ for $k \geqslant N$. For each $k \geqslant N$, choose $y_k \in M_k$ such that $\| x - y_k \| < 2\delta_k(x)$. Define

$$\delta_k = \frac{\| y_k \| - \| x \|}{\| x \|} \qquad \text{for} \quad k \geqslant N.$$

Clearly,

$$| \delta_k | \leqslant \frac{\| y_k - x \|}{\| x \|} < \frac{2\delta_k(x)}{\| x \|} < 1.$$

Set $m_k = (1 + \delta_k)^{-1} y_k$ for every $k \geqslant N$. Then $m_k \in M_k$, $\| m_k \| = \| x \|$, and

$$\begin{aligned} \| x_k - m \| &= \| x - y_k + \delta_k(1 + \delta_k)^{-1} y_k \| \\ &\leqslant \| x - y_k \| + | \delta_k| \, \|(1 + \delta_k)^{-1} y_k \| \\ &< 2\delta_k(x) + | \delta_k | \, \| x \| < 4\delta_k(x). \end{aligned} \qquad \text{Q.E.D.}$$

We observe next that one has a SAIN result with the same bounds (up to a constant) whenever one has a weak SAIN result:

THEOREM 4.2. *Let x be in X and suppose that for each $k \geqslant N_1$ there is an $s_k \in M_k$ for which $x_i^*(s_k) = x_i^*(x)$ $(i = 1,..., n)$ and $\| s_k \| \leqslant \| x \|$. Then there are constants C and N such that for every $k \geqslant N$ there exists $m_k \in M_k$ satisfying*

(1) $x_i^*(m_k) = x_i^*(x)$ $(i = 1,..., n)$

(2) $\| m_k \| = \| x \|$

(3) $\| x - m_k \| \leqslant C \| x - s_k \|$.

Proof. If $\| s_k \| = \| x \|$ for every k, take $m_k = s_k$. Thus we may assume $\| s_k \| < \| x \|$ for some k. Choose $x_0^* \in X^*$, $\| x_0^* \| = 1$, so that $x_0^*(x) = \| x \|$. If $x_0^* = \sum_{i=1}^{n} \alpha_i x_i^*$ for some scalars α_i, then

$$\| x \| = x_0^*(x) = \sum_{i=1}^{n} \alpha_i x_i^*(x) = \sum_{i=1}^{n} \alpha_i x_i^*(s_k) = x_0^*(s_k) \leqslant \| s_k \|,$$

which is impossible. Hence the set $\{x_0^*, x_1^*,..., x_n^*\}$ is linearly independent. Thus we may choose $m \in M$ such that $x_0^*(m) = 1$ and $x_i^*(m) = 0$ $(i = 1,..., n)$. Choose $N \geqslant N_1$ so that $m \in M_N$. For each $k \geqslant N$ choose $\alpha_k \geqslant 0$ such that $\| s_k + \alpha_k m \| = \| x \|$. Setting $m_k = s_k + \alpha_k m$, it follows that $m_k \in M_k$, $\| m_k \| = \| x \|$, and $x_i^*(m_k) = x_i^*(x)$ $(i = 1,..., n)$. Also,

$$x^*(s_k) + \alpha_k = x_0^*(s_k + \alpha_k m) \leqslant \| s_k + \alpha_k m \| = \| x \| = x_0^*(x)$$

implies

$$\alpha_k \leqslant x_0^*(x) - x_0^*(s_k) \leqslant \| x - s_k \|.$$

Hence

$$\| x - m_k \| \leqslant \| x - s_k \| + \alpha_k \| m \| \leqslant (1 + \| m \|) \| x - s_k \|.$$

Taking $C = 1 + \| m \|$ completes the proof. Q.E.D.

As a corollary to Theorem 4.2 above we have that "Weak SAIN" approximation is equivalent to "SAIN" approximation in a Jackson Theorem (rate of approximation) sense. If we combine Theorem 3.2 with Theorem 4.2 above we also get the following:

COROLLARY 4.3. *Suppose there is an $m \in M$ such that $\| m \| < \| x \|$ and $x_i^*(m) = x_i^*(x)$ $(i = 1,..., n)$. Then there exist a constant C and a positive integer N such that for every $k \geqslant N$ there is an $m_k \in M_k$ satisfying*

(1) $x_i^*(m_k) = x_i^*(x)$ $(i = 1,..., n)$

(2) $\| m_k \| = \| x \|$

(3) $\| x - m_k \| \leqslant C\delta_k(x)$.

On spaces $C(T)$, T compact Hausdorff, it is known [4] that one has SAIN if M is a dense subalgebra of $C(T)$ and the bounded linear functionals are all point evaluations, while one need not have SAIN if the bounded linear functionals are not all point evaluations, even if $T = [a, b]$ and $M = \mathscr{P}$ (see example 3.4 above or [4]). We will thus assume the x_i^* to be point evaluations, $x_i^* = e_{t_i}$, $t_i \in T$, for each $i = 1,..., n$ henceforth. We will also require some additional hypotheses on M to insure that one has SAIN. Since we are interested in a Jackson Theorem rather than a Weierstrass theorem, it is not unnatural to impose hypotheses on M via conditions on the subspaces M_k. It turns out sufficient for our purposes to require that $1 \in M$ and that $m_k \in M_k$ implies $m_k^2 \in M_{2k}$ for k sufficiently large. Note that the second condition is slightly weaker than requiring M to be a graded algebra, but that with the first condition, it is sufficient to guarantee that the essential results of Section 4 of [4] hold, as one observes by examining the proofs there, and that in particular there holds the following:

LEMMA 4.4. (Deutsch and Morris [4, p. 365, Corollary 4.1]). *Suppose that M is a dense subspace of $C(T)$ containing the constant functions and the square of any of its elements. Then for each $f \in C(T)$, $f \geqslant 0$ on T, each set $\{t_1, ..., t_n\}$ in T, and each $\eta > 0$, there is an $m \in M$, $m \geqslant 0$ on T, satisfying*

(1) $m(t_i) = f(t_i)$ $(i = 1,..., n)$

(2) $\| m \| = \| f \|$

(3) $\| f - m \| < \eta$.

To handle a different case than that for which we will use Lemma 4.4 in the proof of Theorem 4.7 below, we also require the following result, which geometrically is closely allied to Lemma 4.4 itself, and in fact is derived using it.

LEMMA 4.5. *Suppose that M is a dense subspace of $C(T)$ containing the constant functions and the square of any of its elements. Suppose $\{t_i\}_{i=0}^n$ are distinct points of T. Then there exists a closed subset A of T, containing t_0 in its interior, and an $m \in M$ such that*

(1) $m(t) \leqslant 0$ *on* A,

(2) $m(t_0) = 0$,

(3) $0 < m(t) \leqslant 1$ *on* $T \backslash A$,

(4) $m(t_i) = 1$ $(i = 1, \ldots, n)$.

Proof. If $n = 0$, take $m \equiv 0$ and $A = T$. If $n > 0$ consider $\mathscr{M}_k = \{m \in M; m(t_j) = 0$ for $j \neq k$ and there is an open subset B of T containing every t_j, $j \neq k$, on which $-1 < m(t) \leqslant 0$ holds$\}$. Suppose $m_1, m_2 \in \mathscr{M}_k$. Then $m_1 + m_2 \in M$, since M is a (linear) subspace of $C(T)$. Also, $(m_1 + m_2)(t_j) = m_1(t_j) + m_2(t_j) = 0$, for $j \neq k$. Let B_1, B_2 be open subsets of T containing the t_j $(j \neq k)$ such that $-1 < m_i \leqslant 0$ holds on B_i, $i = 1, 2$, respectively. Let $A_i = m_i^{-1}((-1/2, +\infty))$ as a set function. Since $m_i \in C(T)$, A_i is an open subset of T, and since $m_i(t_j) = 0$ for $j \neq k$, $t_j \in A_i$ for $i = 1, 2$. Let $B = B_1 \cap B_2 \cap A_1 \cap A_2$. Then B is an open subset of T which contains t_j $(j \neq k)$. Moreover, $-\frac{1}{2} < m_1, m_2 \leqslant 0$ on B, so that $-1 < m_1 + m_2 \leqslant 0$ on B, and thus $m_1 + m_2 \in \mathscr{M}_k$. Now suppose $\alpha \in \mathscr{R}$, $\alpha > 0$. Let $A' = m_1^{-1}((-1/\alpha, +\infty))$. Then A' is an open subset of T, and $t_j \in A'$ for $j \neq k$. Let $B = A' \cap B_1$. Then B is an open subset of T containing t_j $(j \neq k)$. Since $-1/\alpha < m_1 \leqslant 0$ on B, $-1 < \alpha m_1 \leqslant 0$ on B, so $\alpha m_1 \in \mathscr{M}_k$. Thus $\mathscr{M}_k$ forms a convex cone. Furthermore, $m_1 + m_1^2 \in \mathscr{M}_k$ whenever $m_1 \in \mathscr{M}_k$, since for $t \in B_1$,

$$0 \leqslant |m_1(t)| < 1 \qquad \text{implies} \quad |m_1^2(t)| \leqslant |m_1(t)|$$

so that

$$\operatorname{sgn}(m_1 + m_1^2)(t) = \operatorname{sgn}(m_1(t)) = -1,$$

and hence $m_1 + m_1^2 \leqslant 0$ on B_1. But

$$m_1^2 \geqslant 0 \qquad \text{implies} \quad -1 < m_1 \leqslant m_1^2 + m_1$$

on B_1, so since $(m_1 + m_1^2)(t_j) = m_1(t_j) + m_1^2(t_j) = 0$ for $j \neq k$, $m_1 + m_1^2 \in \mathscr{M}_k$.

By Urysohn's Lemma, there is a $g_k \in C(t)$ such that $0 \leqslant g_k \leqslant 2$ on T, $g_k(t_k) = 2$, and $g_k(t_j) = 0$, for $j \neq k$. By Lemma 4.4, there is an $r_k \in M$ such that $0 \leqslant r_k \leqslant 2$ on T, $r_k(t_k) = 2$, and $r_k(t_j) = 0$ for $j \neq k$. Then $-r_k \in \mathscr{M}_k$, implying $-r_k + r_k^2 \in \mathscr{M}_k$. Let $m_k = (-r_k + r_k^2)/2$. Then $m_k \in \mathscr{M}_k$, $m_k(t_k) = 1$, and $m_k(t) \leqslant 1$ on T. Let $s = \sum_{k=1}^{n} m_k$. Observe that $s \in M \subseteq C(T)$ and s is bounded by n on T. Also $s(t_0) = 0$ while $s(t_j) = 1$ for $j = 1,..., n$. Let B_k be an open subset of T containing t_j ($j \neq k$) for which $-1 < m_k(t) \leqslant 0$. Let $B' = \bigcap_{k=1}^{n} B_k$. Then B' is open in T, contains t_0, and is disjoint from t_j for $j = 1,..., n$. Moreover $m_k \leqslant 0$ on B', so that $s \leqslant 0$ on B' also. Let $m' \in M$ be such that $m'(t_j) = 0$ for every $j = 0, 1,..., n$, $0 \leqslant m' \leqslant 1$ on T, and $\frac{1}{2} \leqslant m'$ on $T \backslash B'$. Choose $\alpha > 0$ so that $s + \alpha m' \leqslant 1$ on T. Let $m = s + \alpha m'$. Then $m \in M$, $m(t_0) = 0$, $m(t_j) = 1$ for $j = 1,..., n$, and $m \leqslant 1$ on T. Let $A = m^{-1}((-\infty, 0))$. Then A is a closed subset of T, contains B', and $0 < m(t) \leqslant 1$ on $T \backslash A$. Since $t_0 \in B'$ open, t_0 is in the interior of A.

Q.E.D.

Putting the two previous lemmas together, we have the following:

LEMMA 4.6. *Suppose that M is a dense subspace of $C(T)$ containing the constant functions and the square of any of its elements. Suppose that $\{t_i\}_{i=0}^{n}$ are distinct points of T and that U is an open neighborhood of t_0 disjoint from t_i for every $i \neq 0$. Then there is an $m \in M$ such that*

(1) $m(t_0) = 1$,

(2) $m(t_i) = 0$ *for* $i \neq 0$,

(3) $m(t) \leqslant 0$ *on* $T \backslash U$,

(4) $m \leqslant 1$ *on* T.

Proof. By Lemma 4.5, there is an $r_i \in M$ for which $r_i(t_i) = 0$, $r_i(t_j) = 1$ for $j \neq i$, $j = 0, 1,..., n$, $r_i(t) \leqslant 0$ in some open neighborhood V_i of t_i, and $r_i \leqslant 1$ on T, for each $i = 1,..., n$. Let $s = (\sum_{i=1}^{n} r_i) - (n - 1)$. Then $s \in M$, $s(t_0) = 1$, $s(t_i) = 0$ for every $i = 1,..., n$, $s(t) \leqslant 0$ in some open neighborhood V containing t_i, for $i \neq 0$, and $s \leqslant 1$ on T. By Urysohn's Lemma, there is a $g \in C(T)$ for which $g(t) \equiv 1$ on $A = T \backslash (U \cup V)$, $g(t_i) = 0$ for all $i = 0, 1,..., n$ and $0 \leqslant g \leqslant 1$ on T. By Lemma 4.4, there is an $r \in M$ for which $r(t_i) = 0$ for every $i = 0, 1,..., n$, $0 \leqslant r \leqslant 1$ on T, and $\| g - r \| < \frac{1}{4}$. But s is bounded on A, so $r > \frac{3}{4}$ on A implies there is an $\alpha > 0$ such that $s - \alpha r \leqslant 0$ on A. Let $m = s - \alpha r$. Then $m(t_0) = 1$, while $m(t_i) = 0$ for each $i = 1,..., n$. Since $-\alpha r \leqslant 0$ on T, $m = s - \alpha r \leqslant s \leqslant 0$ on V, and so $m \leqslant 0$ on $T \backslash U = A \cup V$. Finally $m \leqslant s - \alpha r \leqslant s \leqslant 1$ on T. Q.E.D.

We introduce the following notation to simplify the statement and proof of our principle theorem. For a given set of bounded linear functionals $\{x_i^*\}_{i=1}^n$, we set

$$p = p(f) = |\{x_i^*; x_i^* f = \|f\|\}| \quad \text{and} \quad q = q(f) = |\{x_i^*; x_i^*(f) = -\|f\|\}|.$$

THEOREM 4.7. *Suppose $\{M_k\}_{k=1}^\infty$ is an increasing sequence of closed linear subspaces of $C(T)$ satisfying*

(a) *its union M is dense in $C(T)$,*

(b) *M contains the constant functions,*

(c) *$m_k \in M_k$ implies $m_k^2 \in M_{2k}$ for sufficiently large k.*

Let $t_1, \ldots, t_n$ be n distinct points in T, and let $f \in C(T)$. Then there exist N and C so that for every $k > N$ there is an $m_k \in M_k$ for which

(1) $m_k(t_i) = f(t_i)$ $(i = 1, \ldots, n)$,

(2) $\| m_k \| = \|f\|$,

(3) $\|f - m_k\| \leqslant C\theta_k(f)$,

where

$$\theta_k(f) = \begin{cases} \delta_k(f), & \text{if } p = q = 0, \\ \delta_{[k/2]}((\|f\| - f)^{1/2}), & \text{if } q = 0, \\ \delta_{[k/2]}((\|f\| + f)^{1/2}), & \text{if } p = 0, \\ \min\{\delta_{[k/4]}(((2\|f\|)^{1/2} - (\|f\| - f)^{1/2})^{1/2}), & \\ \delta_{[k/4]}(((2\|f\|)^{1/2} - (\|f\| + f)^{1/2})^{1/2})\}, & \text{otherwise}. \end{cases}$$

Proof. Let N_2 be such that $1 \in M_{N_2}$, and $N_3 \geqslant N_2$ such that $m_k \in M_k$ implies $m_k^2 \in M_{2k}$ for $k \geqslant N_3$. By Theorem 4.2 it is sufficient to prove the weak SAIN result only. If $n = 0$, the result is Theorem 4.1, so assume $n > 0$.

Case I: $p = q = 0$. Then $|f(t_i)| < \|f\|$ for all $i = 1, \ldots, n$, and the result is Theorem 3.5.

Case II: $p = n$. We define the auxiliary function $g \in C(T)$ by $g = (\|f\| - f)^{1/2}$. Then $g(t_i) = (\|f\| - f(t_i))^{1/2} = 0$ for each $i = 1, \ldots, n$. By Case I, there exist C_1 and N_1 such that for every $k \geqslant N_1$ there is an $s_k \in M_k$ for which $s_k(t_i) = g(t_i) (i = 1, \ldots, n)$, $\| s_k \| = \| g \|$, and $\| g - s_k \| \leqslant C_1 \delta_k(g)$ with $C = C_1$. Let $N = \max[2N_1, N_3]$, and suppose $k \geqslant N$. Set $m_{2k} = \|f\| - s_k^2$. Then $m_{2k} \in M_{2k}$, $m_{2k}(t_i) = \|f\|$, and $\| m_{2k} \| \leqslant \|f\|$, since $0 \leqslant s_k^2 \leqslant 2\|f\|$ implies $-\|f\| \leqslant s_k^2 - \|f\| \leqslant \|f\|$. Also,

$$\begin{aligned} \|f - m_{2k}\| &= \|(\|f\| - g^2) - (\|f\| - s_k^2)\| \\ &= \| s_k^2 - g^2 \| \\ &\leqslant \| s_k + g \| \| s_k - g \| \\ &\leqslant 2(2\|f\|)^{1/2} \| g - s_k \| \\ &\leqslant 2(2\|f\|)^{1/2} C_1 \delta_k(g). \end{aligned}$$

If $g \in M_k$, then $\| g - s_k \| < \eta$ implies $\| f - (\| f \| - s_k^2) \| < 2 \| g \| \eta$ so that $f \in M_{2k}$, so by taking N sufficiently large, $f \in M_k$ if $g \in M_k$. If $g \notin M_k$, then

$$\| g - s_k \| < C_1 \delta_k(g) = C_1 \delta_k((\| f \| - f)^{1/2}),$$

implying

$$\| f - m_{2k} \| < 2(2 \| f \|)^{1/2} C_1 \delta_k((\| f \| - f)^{1/2}).$$

Hence, for every $k \geqslant N$, k even, $\| f - m_k \| \leqslant C\delta_{[k/2]}((\| f \| - f)^{1/2})$, while if $k = 2m' + 1 \geqslant N$ is odd, then $[k/2] = m' = [(k-1)/2]$ and $M_{2m'+1} \supseteq M_{2m'}$, so that $\| f - m_k \| \leqslant C\delta_{[k/2]}((\| f \| - f)^{1/2})$ holds for arbitrary $k \geqslant (N + 1)$, by setting $m_{2m'+1} = m_{2m'}$ for any index k which is odd.

Case III: $0 < p < n, q = 0$. Without loss of generality, suppose $f(t_i) = \| f \|$ for $i = 1,..., p$. By Lema 4.4, for each $j = 1,..., n - p$, there is an $r_j \in M$ for which $r_j(t_{p+j}) = 1$, $r_j(t_i) = 0$ $(i \neq p + j)$, and $0 \leqslant r_j \leqslant 1$. Let $N_4 \geqslant N_3$ be such that $r_j \in M_{N_4}$ for all $j = 1,..., n - p$. Let

$$\epsilon = \min[\| f \| - | f(t_j)|; j = p + 1,..., n],$$

and choose pairwise disjoint open sets $\{U_j\}_{j=1}^{n-p}$ such that (1) $t_j \in U_j$, and (2) $| f(t) - f(t_j)| < \epsilon$ for $t \in U_j$. By Urysohn's Lemma there is a $g_j \in C(T)$ such that $g(t_{p+j}) = 1$, $g(t) \equiv 0$ on $T \backslash U_j$, and $0 \leqslant g_j \leqslant 1$ on T. By Lemma 4.6, there is a $q_j \in M$ such that $q_j(t_{p+j}) = 1$, $q_j(t_i) = 0$ for $i \neq p + j$, $q_j \leqslant 0$ on $T \backslash U_j$, and $q_j \leqslant 1$ on T. Let $N_5 \geqslant N_4$ be such that $q_j \in M_{N_5}$ for every $j = 1,..., n - p$. Let $N_6 \geqslant N_5$ be such that $k \geqslant N_6$ implies $C\delta_{[k/2]}((\| f \| - f)^{1/2}) < \epsilon_1$, where

$$\epsilon_1 = \frac{\min\{\| f \| - | \min(f(t))|, \epsilon/3\}}{n \prod_{j=1}^{n-p} \{(1 + \| r_j \|)(1 + \| q_j \|) \| r_j \| \| q_j \|\}}.$$

By case II, there exist C_1 and N_1 such that for every $k \geqslant N_1$ there is an $s_k \in M_k$ for which $s_k(t_i) = f(t_i)$ $(i = 1,..., p)$, $\| s_k \| = \| f \|$, and $\| f - s_k \| \leqslant C_1 \delta_{[k/2]}((\| f \| - f)^{1/2})$. If $s_k(t_{p+1}) > f(t_{p+1})$, choose α_k so that

$$(s_k + \alpha_k r_1)(t_{p+1}) = f(t_{p+1}),$$

with $0 > \alpha_k \geqslant -\| f - s_k \|$. Let $s_k^{(1)} = s_k + \alpha_k r_1$. Then $s_k^{(1)}(t_i) = f(t_i)$ for $i = 1,..., p + 1$,

$$\| s_k^{(1)} \| = \| s_k + \alpha_k r_1 \| \leqslant \| s_k \| = \| f \|,$$

and

$$\| f - s_k^{(1)} \| \leqslant \| f - s_k \| + | a_k | \leqslant 2 \| f - s_k \|.$$

If $s_k(t_{p+1}) < f(t_{p+1})$, choose α_k so that $(s_k + \alpha_k q_1)(t_{p+1}) = f(t_{p+1})$, with $0 < \alpha_k \leqslant \| f - s_k \|$, and let $s_k^{(1)} = s_k + \alpha_1 q_1$. Then $s_k^{(1)}(t_i) = f(t_i)$ for $i = 1,..., p + 1$, $\| s_k^{(1)} \| = \| s_k + \alpha_k q_1 \| \leqslant \| f \|$, and

$$\| f - s_k^{(1)} \| \leqslant \| f - s_k \| + \alpha_k \| q_1 \| \leqslant (1 + \| q_1 \|) \| f - s_k \|.$$

If $s_k(t_{p+1}) = f(t_{p+1})$, let $s_k^{(1)} = s_k$.

At the general step, $1 < j \leqslant n - p$, if $s_k^{(j-1)}(t_{p+j}) > f(t_{p+j})$, choose α_k so that $(s_k^{(j-1)} + \alpha_k r_j)(t_{p+j}) = f(t_{p+j})$ and $0 > \alpha_k \geqslant -\| f - s_k^{(j-1)} \|$. Then $\alpha_k \geqslant -\| f - s_k^{(j-1)} \| \geqslant -2^{j-1} \prod_{i=1}^{j-1} (1 + \| q_i \|) \| f - s_k \|$, by the inductive step. Set $s_k^{(j)} = s_k^{(j-1)} + \alpha_k r_j$. Then $s_k^{(j)}(t_{p+j}) = f(t_{p+j})$, while

$$s_k^{(j)}(t_i) = s_k^{(j-1)}(t_i) = f(t_i) \qquad \text{for} \quad i = 1,..., p + j - 1,$$

by inductive hypothesis again. Also, $\alpha_k r_j \leqslant 0$ implies $s_k^{(j)} \leqslant s_k^{(j-1)} \leqslant \| f \|$ by the inductive step, while

$$\begin{aligned} \alpha_k r_j &\geqslant -2^{j-1} \prod_{i=1}^{n-p} \{(1 + \| q_i \|)\} \epsilon_1 \\ &\geqslant - \frac{\| f \| - | \min(f) |}{n 2^{n-p-j} \prod_{i=j}^{n-p} \{(1 + \| q_i \|)\}} \\ &\geqslant - \frac{\| f \| - | \min(f) |}{n} \end{aligned}$$

while by the inductive step

$$s_k^{(j-1)} \geqslant -\| f \| + (n - j)(\| f \| - | \min(f) |)/n \geqslant -\| f \|,$$

so that

$$s_k^{(j)} \geqslant -\| f \| + (n - j - 1)(\| f \| - | \min(f) |)/n \geqslant -\| f \|,$$

and hence $\| s_k^{(j)} \| \leqslant \| f \|$. Finally

$$\begin{aligned} \| f - s_k^{(j)} \| &\leqslant \| f - s_k^{(j-1)} \| + | \alpha_k | \\ &\leqslant 2 \| f - s_k^{(j-1)} \| \\ &\leqslant 2 \cdot 2^{j-1} \prod_{i=1}^{j-1} [(1 + \| q_i \|)] \| f - s_k \| \\ &\leqslant 2^j \prod_{i=1}^{j} [(1 + \| q_i \|)] \| f - s_k \| \end{aligned}$$

If $s_k^{(j-1)}(t_{p+j}) < f(t_{p+j})$, choose α_k so that $(s_k^{(j-1)} + \alpha_k q_j)(t_{p+j}) = f(t_{p+j})$ and $0 < \alpha_k \leqslant \| f - s_k^{(j-1)} \|$; and set $s_k^{(j)} = s_k^{(j-1)} + \alpha_k q_j$. Then $s_k^{(j)}(t_i) = f(t_i)$ for $i = 1,..., j$. Since $\alpha_k q_j \leqslant 0$ on $T \backslash U_j$, $s_k^{(j)} \leqslant s_k^{(j-1)} \leqslant \| f \|$ on $T \backslash U_j$. If $t \in U_j$, then

$$\alpha_k q_j \leqslant 2^{j-1} \prod_{i=1}^{j-1} \{(1 + \| q_i \|)\} \epsilon_1 \leqslant \epsilon/3,$$

while $s_k^{(j-1)} \leqslant s_k$, by inductive hypothesis, since the U_j are disjoint, and

$$s_k \leqslant f + \| f - s_k \| \leqslant (f(t_{p+j}) + \epsilon/3) + \epsilon/3,$$

by the uniform continuity of f, so that

$$s_k^{(j)} \leqslant f(t_{p+j}) + \epsilon \leqslant \| f \|$$

on U_j, and thus on all of T itself by the above. Moreover, if $h > j$, then $s_k^{(j-1)} \leqslant s_k$ on U_h, by the inductive hypothesis, so that $s_k^{(j)} \leqslant s_k$ on U_h also. On the other hand,

$$s_k^{(j)} \geqslant -\| f \| + (n - j - 1)(\| f \| - | \min(f) |)/n \geqslant -\| f \|$$

as above, which implies $\| s_k^{(j)} \| \leqslant \| f \|$. Finally

$$\begin{aligned} \| f - s_k^{(j)} \| &\leqslant \| f - s_k^{(j-1)} \| + \alpha_k \| q_j \| \\ &\leqslant (1 + \| q_j \|) \| f - s_k^{(j-1)} \| \\ &\leqslant 2^j \prod_{i=1}^{j} \{(1 + \| q_i \|)\} \| f - s_k \|. \end{aligned}$$

We now take $N = \max[N_1, N_6]$, and let $m_k + s_k^{(n-p)}$. Then, for all $k \geqslant N$, $m_k \in M_k$, $m_k(t_i) = f(t_i)$ for $i = 1,..., p + (n - p) = n$,

$$\| m_k \| \leqslant \| f \|,$$

and with

$$\| f - m_k \| \leqslant 2^{n-p} \prod_{i=1}^{n-p} \{(1 + \| q_i \|)\} \| f - s_k \| \leqslant C \delta_{[k/2]}((\| f \| - f)^{1/2})$$

$$C = C_1 2^{n-p} \prod_{i=1}^{n-p} \{(1 + \| q_i \|)\}.$$

Case IV: $q = n$. Let $h = -f$ and apply Case II.

Case V: $0 < q < n$, $p = 0$. Let $h = -f$ and apply Case III.

Case VI: $0 < p < n$, $0 < q < n$. Without loss of generality, suppose that $f(t_i) = \|f\|$ for $i = 1,..., p$, and that $f(t_i) = -\|f\|$ for $i = p + 1,..., p + q$. We use the auxiliary function $g \in C(T)$ defined by $g = (\|f\| + f)^{1/2}$. Since

$$g(t_i) = (2\|f\|)^{1/2} = \|g\| \quad \text{for} \quad i = 1,..., p$$

while

$$g(t_i) = 0 \quad \text{for} \quad i = p + 2,..., p + q,$$

and

$$0 < g(t_i) < \|g\| \quad \text{for} \quad i = p + q + 1,..., n,$$

by Case III there exist C_1 and N_1 such that for every $k \geqslant N_1$ there is an $s_k \in M_k$ for which $s_k(t_i) = g(t_i)$ $(i = 1,..., n)$, $\|s_k\| = \|g\|$, and $\|g - s_k\| \leqslant C_1\delta_{[k/2]}((\|g\| - g)^{1/2})$. Let $N = \max[N_3, 2N_1] + 1$, and suppose $k \geqslant N$. Let $m_{2k} = s_k^2 - \|f\|$. Then $m_{2k} \in M_{2k}$, $m_{2k}(t_i) = (\|f\| + f(t_i)) - \|f\| = f(t_i)$ $(i = 1,..., n)$ and $\|m_{2k}\| \leqslant \|f\|$, since $0 \leqslant s_k^2 \leqslant 2\|f\|$ implies $-\|f\| \leqslant s_k^2 - \|f\| \leqslant \|f\|$. Finally

$$\begin{aligned}\|f - m_{2k}\| &= \|(g^2 - \|f\|) - (s_k^2 - \|f\|)\| \leqslant \|g + s_k\|\|g - s_k\| \\ &\leqslant 2(2\|f\|)^{1/2} C_1\delta_{[k/2]}((\|g\| - g)^{1/2}).\end{aligned}$$

If $g \in M_k$, then $f \in M_k$, and done. Otherwise,

$$C_1\delta_{[k/2]}((\|g\| - g)^{1/2}) = C_1\delta_{[k/2]}(((2\|f\|)^{1/2} - (\|f\| + f)^{1/2})^{1/2}),$$

and take $C = 2(2\|f\|)^{1/2} C_1$.

Using the auxiliary function $g = (\|f\| - f)^{1/2}$, we get the estimate

$$C_1\delta_{[k/2]}((\|g\| - g)^{1/2}) \leqslant C_1\delta_{[k/2]}(((2\|f\|)^{1/2} - (\|f\| - f)^{1/2})^{1/2}).$$

Taking the minimum of these two estimates, and finishing as in Case II, the result follows. Q.E.D.

In particular, on $C[a, b]$ with $M_k = P_k$ polynomials of degree k, in which setting we may apply the classical Jackson Theorem [12] to get the estimate $\delta_k(f) \leqslant 12(1 + (b - a)/2)\omega_f(k^{-1})$ for $k = 1, 2,...$, our principle theorem reduces to the following:

THEOREM 4.8. *Suppose $f \in C[a, b]$, $\{x_i\}_{i=1}^n \subseteq [a, b]$, and σ a constant, $\sigma = \pm 1$. Then there exist C and N so that for all $k \geqslant N$ there is a $p_k \in P_k$ for which*

(1) $p_k(x_i) = f(x_i)$ $(i = 1,..., n)$,

(2) $\|p_k\| = \|f\|$,

(3) $$\|f - p_k\| \leqslant \begin{cases} C\omega_f(k^{-1}), & \textit{if } |f(x_i)| < \|f\| \quad (i = 1,..., n), \\ C\omega_f^{1/2}(k^{-1}), & \textit{if } |f(x_i)| < \|f\| \textit{ or } f(x_i) = \sigma\|f\|, \\ & \qquad i = 1,..., n, \\ C\omega_f^{1/4}(k^{-1}), & \textit{otherwise}, \end{cases}$$

and hence $\|f - p_k\| \leqslant C\omega_f^{1/4}(k^{-1})$.

Proof. We have the estimates

$$\begin{aligned}
\delta_k(f) &\leqslant 12(1 + (b - a)/2)\, \omega_f(k^{-1}), \\
\delta_{[k/2]}((\|f\| \pm f)^{1/2}) &\leqslant 12(1 + (b - a)/2)\, \omega_{(\|f\|\pm f)^{1/2}}([k/2]^{-1}) \\
&\leqslant 24(1 + b - a)/2\omega^{1/2}_{\|f\|\pm f}(k^{-1}) \\
&\leqslant 24(1 + (b - a)/2)\, \omega_f^{1/2}(k^{-1}), \\
\delta_{[k/4]}(((2\|f\|^{1/2}) - (\|f\| \pm f^{1/2})^{1/2}) &\leqslant 48(1 + (b - a)/2)\, \omega^{1/2}_{(2\|f\|)^{1/2} - (\|f\|\pm f)^{1/2}}(k^{-1}) \\
&\leqslant 48(1 + (b - a)/2)\, \omega^{1/2}_{(\|f\|\pm f)^{1/2}}(k^{-1}) \\
&\leqslant 48(1 + b - a)/2)\omega_f^{1/4}(k^{-1}).
\end{aligned}$$

Hence the result follows immediately from Theorem 4.7. Q.E.D.

Clearly, the theorem is valid if the e_{x_i} are replaced by any $x_i{}^*$ in the span of $\{e_{x_1}, ..., e_{x_n}\}$. Hence

THEOREM 4.9. *Suppose* $f \in C[a, b]$, $\{x_i\}_{i=1}^n \subseteq [a, b]$, *and* $y_j{}^* = \sum_{i=1}^n a_i e_{x_i}$, $j = 1,..., m$. *Then there exist* C *and* N *such that* $k \geqslant N$ *implies there is a* $p_k \in P_k$ *for which*

(1) $y_j{}^*(p_k) = y_j{}^*(f)$ $(j = 1,..., m)$,

(2) $\|p_k\| = \|f\|$,

(3) $\|f - p_k\| \leqslant C\omega_f^{1/4}(k^{-1})$.

In Theorem 4.8 we considered arbitrary finite linear combinations of point evaluations on $C[a, b]$. We show in Example 4.9 below that we cannot consider arbitrary infinite linear combinations of point evaluations, however. Example 4.9 also shows that a result obtained by Lambert [10] is best possible in that given any $f \in C[a, b]$ which attains its norm infinitely often, there exists a bounded linear functional for which one does not have SAIN holding.

Example 4.9. Suppose $f \in C[a, b]\backslash\mathscr{P}$ attains its norm at the countably infinite number of points $\{x_i\}_{i=0}^\infty \subseteq [a, b]$. Let $(a_i) \in \ell_i$ be such that $\text{sgn}(a_i) = \text{sgn}(f(x_i))$ for all i. Let $y^* = \sum_{i=0}^\infty a_i e_{x_i}$. Then $\|y^*\| = \sum_{i=0}^\infty |a_i| = \|(a_i)\|_{\ell_1} < \infty$, so that y^* is a bounded linear functional on $C[a, b]$. Also, $y^*(f) = \sum_{i=0}^\infty a_i \,\text{sgn}(a_i)\|f\| = \|y^*\|\,\|f\|$. But, if $p \in \mathscr{P}$ is any polynomial for which $\|p\| \leqslant \|f\|$, then $y^*(p) < \|y^*\|\,\|f\|$, unless $\text{sgn}(a_i) = \sigma$ is constant, $\sigma = \pm 1$ and $p \equiv \sigma\|f\|$. Hence one does not have SAIN.

Remark 4.10. We observe that if we take $T = T^1$ the unit circle and $M_k = T_k$, trigonometric polynomials of degree k, we have analogous results to Theorems 4.8 and 4.9, the statements being identical except for replacing $[a, b]$ by T^1 and P_k by T_k, so that trigonometric approximation is handled exactly as algebraic approximation.

Remark 4.11. In weak norm preservation, our approximating elements satisfy the condition $-\|f\| \leqslant m_k \leqslant \|f\|$. Suppose one replaces weak norm preservation by the condition $a \leqslant m_k \leqslant b$, where $a \leqslant f \leqslant b$. On $C[a, b]$ with polynomials, it is trivial that the same estimates hold, as one need only let $g = f - (b + a)/2$, apply Theorem 4.7 to get p_k' approximating g, and let $p_k = p_k' + (b + a)/2$. However, if one replaces the constants a, b by functions $a(x)$, $b(x)$, it is no longer a triviality but an interesting question which has been considered by V. A. Šmatkov [14].

Acknowledgments

I thank my advisor Professor Bruce L. Chalmers for suggesting this problem, and for his invaluable guidance and encouragement in seeing it through. I also thank Professor Frank R. Deutsch for many helpful suggestions, and in particular for the simplication of several proofs and the generalization of one of the theorems.

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